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Accuracy in Simulations

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Since the actual solution to intertemporal rational expectations models is usually not known, it is useful to have criteria for judging the accuracy of a given numerical solution. In this paper we propose a test for accuracy that is easy to implement and can be applied to a wide class of models without knowledge of the exact solution. We discuss the power of the test by simulating several models with the linear-quadratic approximation and with the method of parameterized expectations. We conclude that the test is powerful.

1. INTRODUCTION

Because of the difficulty in finding analytic solutions to stochastic dynamic models with rational expectations, it is becoming more and more common to resort to numerical methods for solving these models. Recent applications to economics are reviewed in Marcet (1993). Different simulation methods are based on different types of approximation and, therefore, they are subject to approximation error.

Some procedures can, in principle, approximate the solution arbitrarily well. These include procedures that (i) convert continuous variables into discrete ones by use of a grid, for example, the procedures used by Miller (1984), Rust (1987), Tauchen (1986), and Wolpin (1984), and (ii) methods that approximate some unknown function with flexible functional forms of finite elements (for example polynomials or linear interpolation), such as in the methods discussed in Coleman (1991), Marcet (1988) and Judd (1989). Methods of type (i) can get an arbitrarily accurate solution by refining the grid, and methods of type (ii) can get an accurate solution as the number of elements in the function (e.g., the degree of the polynomial) goes to infinity. Nevertheless, computing costs often hold the approximation to a level that has no guarantee of being the 'correct' one.

The issue of accuracy is even more important in methods where the solution cannot be refined. For example, the backwards solution procedure of Sims (1989), Ingram (1990) and Novales (1991); the extended path method of Fair and Taylor (1983) and the linear-quadratic approximation used, for example, by Kydland and Prescott (1982) and Christiano (1990).

In this paper we propose a test for the accuracy of a numerical solution to a rational expectations model. Our test is very easy to implement, it is computationally inexpensive, it can be performed without knowledge of the true solution and it can be used by most solution procedures. We will discuss the principle underlying the test, as well as several ways of implementing it. Versions of the test-statistic proposed in this paper are used in Taylor and Uhlig (1990) to compare solutions to the simple growth model with alternative

methods: their comparisons suggest that those methods that passed the test also gave similar solutions.

The test presented here is powerful enough to discover inaccuracies in several models. In fact, we will present cases where some important characteristics of the solution are fairly accurate, but the accuracy test still rejects the solution. In the case where the solution is rejected, the accuracy test can be used to determine which part of the solution is inaccurate. For example, in Section 4.3 we discuss an example where the real part of the model is accurate, but the monetary part is not.

Our test can be used as guidance for applying a given algorithm, when we have to select functional forms, size of the grid, convergence criterion, etc. For example, it can tell us if we should use a higher-degree polynomial. Or, as another example, it can help in choosing between a linear-quadratic (LQ) and a log-linear-quadratic approximation (log LQ) as in Christiano (1990). Furthermore, when used in conjunction with the parameterized expectations algorithm, the test indicates if some elements of a higher-order polynomial do not have to be introduced, thereby reducing the cost of using high-order polynomials.

The papers by Tauchen (1991), Danthine, Donaldson and Mehra (1991) and Christiano (1990) evaluate the accuracy of a particular algorithm in solving a particular model by comparing an approximate solution with an 'exact' solution found by either analytic methods or by using a very fine grid. This way of checking for accuracy is only indicative, since the conclusion is model-dependent. More importantly, in general we do not have analytical solutions and we cannot spend huge amounts of computing time only to check for accuracy, so that we cannot compare our solutions to an 'exact' one. Our test can be applied without any knowledge of the exact solution.

In Section 2 we present the idea behind the test and discuss how it can be implemented. In Section 3 we illustrate the properties of the accuracy test with a simple example; we will see that the results of the accuracy tests improve as the solution procedure becomes more precise. In Section 4 it is shown, that the results of the accuracy tests indicate that the log LQ approximation is more accurate than the LQ solution, which is the same result obtained by Christiano (1990) with a more expensive procedure for testing accuracy. We also give an economic justification of the better performance of log LQ in that model by arguing that the restrictions imposed by the log LQ approximation make more economic sense. In order to see how the test would help in comparing solutions that can obtain arbitrary accuracy with solutions that cannot, we also compare the solutions obtained by the parameterized expectations approach[1] versus LQ and log LQ.

2. A TEST FOR ACCURACY

Consider an economic model where a set of variables z_t completely describes the economy at time t. Some of the elements in z_t can be exogenous. In stochastic dynamic models with rational expectations, it is often the case that the solution is stationary and ergodic, and that a system of equations of the following type has to be satisfied:

$$f(z_t) = E(\phi(z_{t+1}, z_{t+2}, \ldots) \mid \Omega_t), \tag{2.1}$$

1. This approach has been applied among others, by Marshall (1988), den Haan (1990a, b), Marcet and Singleton (1989) and Marcet and Marimon (1992).

where, given the parameters of the model, $f: R^n \to R^m$ and $\phi: R^n \times R^\infty \to R^m$ are known functions, and where $E(x \mid y)$ represents the conditional expectation of x given y. The information set Ω_t contains a subset of current and past values of z_t. Notice that we can accommodate models with sub-optimal equilibria, private information and inequality constraints. For many stochastic dynamic models one can guarantee that if certain side conditions are satisfied (such as transversality conditions or time-invariant solutions), then, (2.1) determines the solution uniquely.

If the above equation is satisfied, then the residual

$$u_{t+1} = \phi(z_{t+1}, z_{t+2}, \ldots) - f(z_t) \tag{2.2}$$

satisfies

$$E[u_{t+1} \otimes h(x_t)] = 0, \tag{2.3}$$

for any k-dimensional vector x_t that belongs to the information set Ω_t, and any function $h: R^k \to R^q$. The idea for the test is to see if equation (2.3) is close to being satisfied for the simulated series $\{\bar{z}_t\}$ obtained with a certain numerical algorithm. (Throughout the paper we use the bar to denote simulated series.)

In general this is a challenging test, because most solution procedures do not enforce (2.3).[2] It is also a meaningful test because, if we could check (2.3) for *any* function $h(\cdot)$, if (2.1) contained all the equations that determine the solution of the model, and x_t is a vector of sufficient state variables, in effect we would have shown that our solution is exact, since the conditional expectation is the only function having this property. When equations (2.1) come from the maximization problem of the agents, testing (2.3) can be interpreted as testing if the derivative of the solution is zero along the directions given by the function $h(\cdot)$.[3]

The accuracy test consists of obtaining long simulations of the process and calculating

$$B_T \equiv \frac{\sum^T \bar{u}_{t+1} \otimes h(\bar{x}_t)}{T},$$

where $\bar{u}_t$ and $h(\bar{x}_t)$ are calculated with the simulated $\bar{z}_t$, and checking if B_T is close to zero. Clearly, if the solution were exact B_T converges to zero almost surely as T goes to infinity.

Obviously, B_T will never be exactly equal to zero because of sampling error. The question that arises is how to decide whether B_T is *significantly* different from zero. We have to be careful because B_T could be made arbitrarily small by, for example, taking a function $h(\cdot)$ with sufficiently small function values. One way of avoiding this problem is to use the distribution of some test-statistic related to B_T under the null hypothesis that the solution is accurate. To be precise, we use $TB'_T A_T^{-1} B_T$ as our test-statistic, where A_T is some consistent estimate of the matrix

$$S_w = \sum_{i=-\infty}^{\infty} E[[u_{t+1} \otimes h(x_t)] \cdot [u_{t+1-i} \otimes h(x_{t-i})]'].$$

2. This is not true of the backwards solution procedure, that constructs u_t in a way that guarantees (2.3) to hold for any $h(\cdot)$. To do a similar test for that procedure one would have to test if the innovations to the exogenous shocks that the backwards solution recovers are orthogonal to lagged variables.

3. It is of course impossible to check whether (2.3) is satisfied for any function $h(\cdot)$; at the end of this section, however, we discuss a procedure with which it is possible to avoid the problem of checking (2.3) in a predetermined dimension.

There are many choices for A_T. The simplest case is when $\phi(z_{t+1}, z_{t+2}, \ldots)$ depends only on z_{t+1} and $m=1$; then we can use

$$A_T=\frac{\sum^T \vec{u}_{t+1}^2 h(\bar{x}_t)h(\bar{x}_t)'}{T}.$$

Consistent estimators of S_w in more general cases are described in the GMM literature; see, for example, Hansen (1982), Newey and West (1987) and Christiano and den Haan (1993).[4]

The test-statistic satisfies:

Proposition 1. *If $\{z_t\}$ is stationary and ergodic, the numerical solution is exact, S_w is finite and invertible, then*

$$TB'_T A_T^{-1} B_T \xrightarrow{D} \chi^2_{qm} \quad \text{as } T\to\infty.$$

Proof. The process $\{u_{t+1}\otimes h(x_t)\}$ is a function of a stationary and ergodic process, so it is itself stationary and ergodic. We have that $E(u_{t-1}\otimes h(x_t)\,|\,z_t, z_{t-1}, \ldots)=0$ and $S_w<\infty$, so that all the conditions of Theorem 5.15 in White (1984) are satisfied, and

$$T^{1/2}B_T=(1/T)^{-1/2}\sum^T u_{t-1}\otimes h(x_t) \xrightarrow{D} N(O, S_w) \quad \text{as } T\to\infty.$$

Given a positive semi-definite matrix M, we let $M^{-1/2}$ be some uniquely determined decomposition satisfying $M^{-1/2}M^{-1/2'}=M^{-1}$. Since S_w is invertible, the mapping f defined by $f(M)=M^{-1/2}$ is continuous at S_w and $A_T^{-1/2}$ converges in probability to $S_w^{-1/2}$, so that $T^{1/2}A_T^{-1/2}B_T \to^D N(O, I)$. Since $A_T^{-1/2}B_T$ is a qm-dimensional vector, the proposition folllows. ||

The above proposition mimics the test of over-identifying restrictions of Hansen (1982). The proof is very similar and is offered only for completeness. One important difference is that, in our case, the parameters that generate the observations are known with certainty so that there is no loss of degrees of freedom in the χ^2 distribution due to the number of estimated parameters.

This result suggests the following:

Accuracy test. Obtain a large number of observations by simulating the model for a realization of the exogenous process. If the value of $TB_T A_T^{-1} B'_T$ belongs to the lower or upper critical region of a χ^2_{qm} distribution, there is evidence against the accuracy of the solution.

Note that this test can be implemented without any knowledge of the analytic form of the true solution, so it has wide applicability. Also note that, for most solution procedures, simulating the model for a realization of the exogenous process is computationally inexpensive and so is calculating $TB_T A_T^{-1} B'_T$. As with any other test-statistic it is possible that an accurate solution is rejected (Type I error) or an inaccurate solution is not rejected (Type II error). But, since the test can be performed at such low costs, Type I error can

4. Some of these estimates are designed especially to guarantee positive-semidefiniteness of the estimator. Since we are using many observations and the true model, it is less likely that the estimator will be non-positive semidefinite even if the estimator is not so by construction.

be eliminated by repeating it for different realizations of the exogenous process and reporting the percentage in the upper and lower critical 5% of a χ^2_{qm} distribution. This to convince the reader that not rejecting the null-hypothesis is not due to a lucky draw. For the solutions that we tested we repeated the accuracy test 250 and 500 times and we found very similar answers for both number of repetitions.

To implement the test the researcher has to choose the number of observations (T) and the function $h(\cdot)$; these choices will be discussed next. We know that any given numerical solution will fail the accuracy test for sufficiently large T. The reason is that, since any numerical solution is only an approximation, (2.3) is never exactly satisfied by the simulated series, and the test-statistic will discover this if enough observations are used. Therefore, the choice of T governs the stringency of the test: the higher T the harder it is for a given solution to be deemed accurate. One possible criticism to this test is that it makes no sense to ask-whether B_T is close to zero, since we know that this will not be true for T high enough, but this could be said of most accuracy criteria used traditionally in numerical computations. For example, assume we have a function $F(x)$ and we find its fixed point $\bar{x}=F(\bar{x})$ numerically. The usual way of checking if this fixed point is correct is by looking at the number of digits that $F(\bar{x})$ and $\bar{x}$ have in common, even though for a given $\bar{x}$ found numerically, there is always a criterion stringent enough (a high enough number of digits) that will deem $\bar{x}$ inaccurate.

Therefore, in our case, T can be interpreted as a measure of how stringent the criterion is: if the solution passes the test even for a very large T, this is evidence that the solution is very accurate. The user of the accuracy test has to be careful to report the T that he used, just as a researcher calculating a fixed point numerically should report the number of digits of accuracy. In the next section we show that at least for the growth model it is possible to come up with a solution that is so accurate that the test passes even in enormous samples. If one wants to compare the data generated with a numerical solution with an actual series, then we suggest choosing T substantially bigger than the length of the empirical series; in this way, one can ensure that the numerical error is small compared with the inevitable sampling error contained in real data. In the following sections we used 3000 observations, which is around 20 times as big as a typical post-war series of quarterly data.

It is clear that $h(\cdot)$ can be chosen in an infinite number of ways. In the examples we tried we found little sensitivity to this choice, but this may be problem-specific. In principle we would want to choose $h(\cdot)$ in such a way as to maximize the power of the test statistic. An idea from Lee, White and Granger (1989) can be used to increase the power of the test statistic; these authors develop a test for neglected nonlinearity in time-series models by choosing the dimension in which to test the null hypothesis randomly. A simple example is the following: suppose that x_t contains one element, e.g. the capital stock k_t; letting ζ be a random variable, we can choose $h(x_t)=k_t^{\zeta}$ (here, ζ is a power). In the next few sections we show that the test is already very powerful even in identifying small inaccuracies, so we do not pursue this idea of randomizing $h(\cdot)$ in the present paper.

Since the test is so powerful it is possible that a numerical solution fails the accuracy test, even though some important characteristics of the solution, like second moments, are accurately simulated.

2.1. *Applying the test to the parameterized expectations method (PEA)*

The method of parameterized expectations, described in Marcet (1988) and den Haan and Marcet (1990), substitutes the conditional expectation in the right-hand side of (2.1) by

a particular function $\psi(\beta_f, s_t)$, where s_t represents the state variables in the model. Here ψ, β_f and s_t are chosen so that the function $\psi(\beta_f, \cdot)$ is close to the conditional expectation in (2.1). Among other things, this means that β_f solves the following minimization problem:

$$\min_{\beta} (1/T^*)\sum^{T^*} [\phi(\bar{z}_{t+1}) - \psi(\beta, \bar{s}_t)]^2, \tag{2.4}$$

where $\bar{z}_t$ and $\bar{s}_t$ are simulated by substituting the conditional expectation in (2.1) with the parameterized expectation $\psi(\beta_f, \bar{s}_t)$. From the first-order conditions of (2.4) we have that

$$\frac{1}{T^*}\sum^{T^*} \bar{u}_{t+1}\frac{\partial\psi(\beta_f, \bar{s}_t)}{\partial\beta} = 0 \tag{2.5}$$

by construction. Hence, for the particular choice $h \equiv \partial\psi(\beta_f, \cdot)/\partial\beta$, for the particular realization used in finding β_f, and for the particular sample size T^* (which need not be equal to T), the parameterized expectations solution makes B_T exactly equal to zero. The accuracy test would always be in the lower tail (but never in the upper tail), and it would always reject the solution obtained with this method.

Therefore, in order to use the accuracy tests on solutions obtained with this method, we have to choose a function h that is different from $\partial\psi/\partial\beta$; also, in calculating the test statistics one should use a realization of the stochastic exogenous shocks that is different from the one used in calculating β_f.

Finally, the accuracy test can be used for guidance as to what type of function ψ should be used in PE. In this solution procedure, as in all finite element procedures, arbitrary accuracy can be obtained, in principle, by increasing the degree of the polynomial.[5] In practice, however, increasing the degree of the polynomial is costly, since higher degree polynomials entail more terms and the number of coefficients involved increases quite fast. For example, with three state variables, going from a first-degree to a second-degree polynomial involves going from a polynomial of four coefficients to one with ten coefficients, since all cross-products of the state variables belong to the second-degree polynomial. In practice, however, many of these second-degree terms do not need to be included; the reason is that many of them are nearly redundant and they do not add to the predictive power of the parameterized expectation.

Clearly, if this happens, it just means that some of these terms can be excluded from the polynomial approximation with no loss of accuracy. More precisely, in going to a solution with a polynomial of degree ν to a polynomial of degree $\nu+1$, we would take the solution with degree ν, make $h(\cdot)$ consist of each element of order $\nu+1$, and exclude from the larger polynomial those elements that passed the accuracy test. In this way we can save a lot of computing power and, in principle, the solution should not lose any accuracy, since the excluded terms do not contribute to the predictive power of the expectation and they do not affect the simulation.

3. ILLUSTRATING THE PROPERTIES OF THE ACCURACY TEST

In this section we illustrate some of the properties of the accuracy test. For this purpose the simple growth model is solved with the method of parameterized expectations. We choose to work with a high standard deviation of the technology shock to make the problem more challenging. We will see that the results of the accuracy tests improve as

5. A formal proof of this approximation result is available for the PEA. See Marcet and Marshall (1992).

TABLE 3.1

Accuracy of the growth model ($T=3000$)

	Lower 5%	Upper 5%
First-order polynomial	0·6%	29·6%
Second-order polynomial	4·8%	6·4%
Third-order polynomial	4·6%	5·6%

the solution procedure becomes more precise. Moreover, it is possible to construct a solution in which the errors of the numerical solution are so small, that the results of the accuracy tests remain good, even when the sample size T is enormous.

The representative agent solves

$$\max_{\{c_t,k_t\}} E_0 \sum_{t=0}^{\infty} \delta^t \frac{c_t^{1-\tau}}{1-\tau}$$

$$\text{s.t.} \quad c_t + k_t = \theta_t k_{t-1}^{\alpha} + \mu k_{t-1} \tag{3.1}$$

$$\log \theta_t = \rho \log \theta_{t+1} + \varepsilon_t. \tag{3.2}$$

Here, ε_t is i.i.d., has a $N(0, \sigma^2)$ distribution and k_{-1} and θ_0 are given. A list of variables is given at the end of the Appendix. The parameter values used are $\tau = 0{\cdot}50$, $\alpha = 0{\cdot}33$, $\mu = 1{\cdot}00$, $\rho = 0{\cdot}95$, $\sigma = 0{\cdot}10$, $\delta = 0{\cdot}95$.

Three solutions to this model are obtained by parameterizing the conditional expectation in the Euler equation. The solutions differ in the order of the polynomial used for the parameterization. Details are given in the Appendix. The prediction error corresponding to the first-order condition of capital is given by

$$u_t = \delta c_{t+1}^{-\tau}(\alpha k_t^{\alpha-1}\theta_{t+1} + \mu) - c_t^{-\tau}; \tag{3.3}$$

this equation will play the role of (2.2) in the accuracy tests. For all three solutions the growth model is simulated 500 times with 3000 observations. We use as instruments $h(x_t) = [1, k_t, k_{t-1}, k_{t-2}, \theta_t, \theta_{t-1}, \theta_{t-2}]$, so that the test statistic has a χ_7^2 distribution. In Table 3.1 we report the percentage of draws that are in the lower and upper 5% tails.

We see that the first-order parameterization is clearly inaccurate, since too many times the test statistic is higher than the critical 95% value and it is not lower than the critical 5% often enough. The results for the second- and third-order polynomial, however, are very close to the theoretical 5%. In Figure 3.1 the whole distribution of the test statistic is given. The conclusions that can be drawn from the graph are the same as the conclusions from the table. That is, only the first-order approximation is not close to the distribution of the true solution.

The sensitivity of the accuracy test to the number of observations used in calculating the test statistic is investigated by increasing the number of observations for the solution of the third-order polynomial. Only a slight increase in the number of draws in the higher 5% region is observed. No decline in the number of draws in the lower 5% could be detected. Even with enormous samples of 20,000 observations, which make for a very stringent test (see discussion in Section 2) we only get 9·8% in the upper 5% and 4·6% in the lower 5% region.

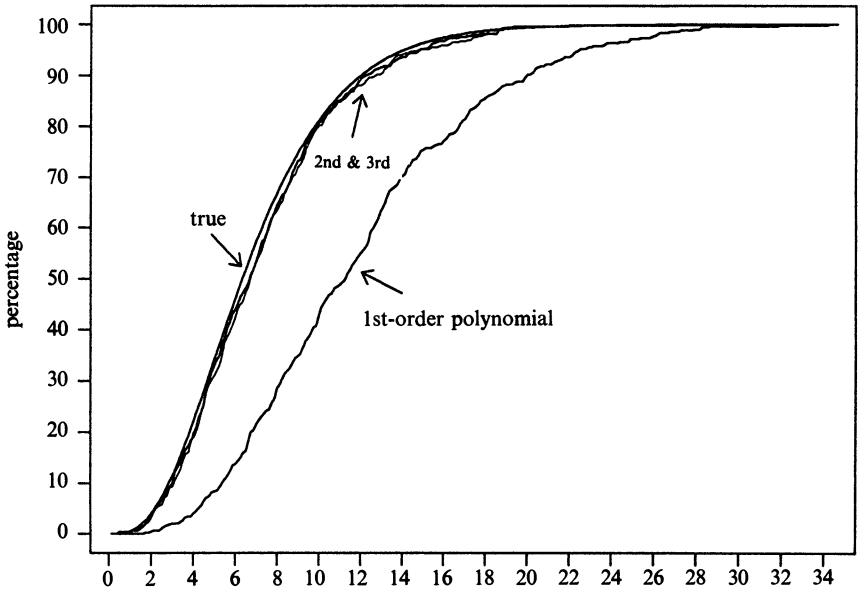

FIGURE 3.1
Accuracy for the growth model

4. USING THE ACCURACY TEST TO CHOOSE BETWEEN ALTERNATIVE SOLUTIONS

4.1. *Choosing Linear-Quadratic approximations in the simple growth model*

In Taylor and Uhlig (1990) it is shown that, even for the simple growth model with a low standard deviation, different solution techniques may display different characteristics of the model. It is thus certainly not the case that computation methods can in general be used interchangeably. Also, given a method, choices have to be made about the functional form, the grid, the convergence criterion, etc. The accuracy tests can be used as a device in making these choices.

In this section we compare the results of the accuracy test for LQ and log LQ in the simple growth model. Christiano (1990) shows that except in states that occur with a low probability log LQ is more accurate than LQ. He obtains this conclusion by comparing the solutions to the solution obtained by value function iteration, which is very accurate (but also very expensive).

Our test confirms the results in Christiano (1990) on the high degree of accuracy of the log LQ approximation, without the need of performing value function iterations. Note that the standard accuracy test automatically puts less weight on states that occur less often in the steady-state distribution.[6] In addition we provide an economic explanation of why log LQ is a better approximation than LQ: both approximations impose restrictions on the means of the variables, but the restrictions imposed by the log LQ technique are shown to be more realistic. For a user of LQ approximations, our accuracy test would be useful in choosing between these two alternatives.

6. If a researcher is interested in the accuracy of his solution for a specific range of the state variables, he can always condition on the state variables being in that specific set.

TABLE 4.1

Accuracy of the solution techniques ($T=3000$)

	$\tau=0\cdot5$		$\tau=3\cdot0$	
	Lower 5%	Upper 5%	Lower 5%	Upper 5%
$\sigma=0\cdot01$				
LQ	0·4%	54·6%	5·2%	10·4%
log LQ	4·0%	8·8%	4·6%	4·6%
PE	4·4%	5·2%	4·8%	3·8%
$\sigma=0\cdot02$				
LQ	0·0%	94·4%	2·8%	36·8%
log LQ	2·2%	28·6%	4·6%	7·8%
PE	3·8%	5·0%	5·4%	4·4%
$\sigma=0\cdot03$				
Lq	0·0%	99·8%	0·4%	64·0%
log LQ	0·6%	49·4%	2·2%	13·8%
PE	5·4%	5·0%	4·8%	5·8%

4.2. *Accuracy comparisons between LQ and parameterized expectations*

Again we use the growth model discussed in the last section. The model is solved with the method of parameterized expectations and with the log LQ and the LQ solution method. We solve the growth model with $\tau=0\cdot50$ and $3\cdot00$, $\alpha=0\cdot33$, $\mu=0\cdot975$, $\rho=0\cdot95$, $\sigma=0\cdot01$, $0\cdot02$ and $0\cdot03$, $\delta=0\cdot99$.

Details about the solutions are again given in the Appendix. For the low-variance case, we cannot use all seven instruments from the last section to do the accuracy tests, since for some parameter values we obtain a nearly singular weighting matrix S_w. This only means that some elements from $h(\cdot)$ are redundant and they can be omitted from the test with no loss of power; therefore, we use the constant as the only instrument.[7] In Table 4.1 the results of the accuracy tests are given. Again, the tests use 500 draws of 3000 observations.

In Table 4.1 we see that the accuracy test would tell us that the LQ approximation is clearly inaccurate for five out of the six parameter cases, while the log LQ approximation is clearly more accurate, although its results are not satisfactory for all parameter values. For the PE solution, a second-order polynomial was chosen in order to pass the accuracy tests for the high standard deviation. One explanation provided in Christiano (1990) of why the log LQ solution method performs better than the LQ solution, is that for the special case of the model with complete depreciation and log utility, the rational expectations policy functions are exactly log LQ. We want to discuss another explanation; the policy functions for the LQ and log LQ approximation are given, respectively, by

$$\bar{k}_t = a_1 + a_2\bar{k}_{t-1} + a_3 \log(\theta_t) \tag{4.1}$$

$$\log(\bar{k}_t) = b_1 + b_2 \log(\bar{k}_{t-1}) + b_3 \log(\theta_t), \tag{4.2}$$

where a_2, a_3, b_2 and b_3 are positive (again, we use bars to denote the approximated series). Values for a's and b's are given in the Appendix. If we take the unconditional expectation,

7. Note that, since we use an exponential polynomial (see equation A.3 in the appendix) the derivative of the parameterized expectation with respect to the constant term is given by $\partial\psi/\partial\beta_1 = \psi/\beta_1$, which is not equal to one. Therefore, even if $h(\cdot)$ is constant, it is not equal to the derivative of ψ and the problems described at the end of Section 2 are avoided.

TABLE 4.2

The mean capital stock for different values of σ

	$\tau = 0{\cdot}5$		
	$\sigma = 0{\cdot}01$	$\sigma = 0{\cdot}02$	$\sigma = 0{\cdot}03$
LQ	28·353	28·357	28·361
	(0·007)	(0·013)	(0·020)
log LQ	28·371	28·432	28·529
	(0·007)	(0·013)	(0·020)
PE	28·385	28·479	28·635
	(0·007)	(0·013)	(0·020)
	$\tau = 3{\cdot}0$		
	$\sigma = 0{\cdot}01$	$\sigma = 0{\cdot}02$	$\sigma = 0{\cdot}03$
LQ	28·360	28·371	28·382
	(0·011)	(0·022)	(0·032)
log LQ	28·411	28·577	28·848
	(0·011)	(0·022)	(0·033)
PE	28·526	29·007	29·790
	(0·011)	(0·022)	(0·033)

Note: The approximate standard errors are given in parentheses. The means are for a sample of 25,000 observations.

then we get for the LQ capital stock

$$E(\bar{k}_t) = \frac{a_1}{1 - a_2} \tag{4.3}$$

$$E(\log \bar{k}_t) = \frac{b_1}{1 - b_2}, \tag{4.4}$$

where the expectation is taken with respect to the steady-state distribution.

In the simple growth model, there are two reasons why, in the true solution, the mean of the capital stock becomes higher with an increase in uncertainty. First, if σ_ε^2 goes up, the mean of θ_t goes up, so that average productivity increases; second, since the representative agent uses capital as an asset for insuring against periods of low productivity, if the uncertainty increases he needs more insurance and a higher capital average. On the other hand, the log LQ solution forces the mean of capital to go up: since the exponential function is convex, an increase in the variance of $\log \bar{k}_t$ will cause the mean of $\bar{k}_t$ to go up. The point we want to drive home is that the log LQ solution, by construction, causes the mean of the capital stock to increase, which in this model happens to be the right direction in which the mean of capital should move. This suggests that in a model where the true solution had a *lower* mean of the state variable when uncertainty increased, plain LQ may be better than log LQ.

Tables 4.2 and 4.3 illustrate the (in)dependence between the mean capital value and the standard deviation for the three solution techniques. Although the mean of the log LQ capital stock rises with σ, it does not rise as much as the PE capital stock. First, consider the case where $\tau = 3{\cdot}0$. If we increase σ from 0·01 to 0·02, for instance, then we find an increase in the log LQ capital stock of 0·58% and an increase in the PE capital stock of 1·70%. This is consistent with Christiano (1990) who also finds that the log LQ average capital stock is between the average capital stock of the LQ and the value function iteration solution.

TABLE 4.3

The increase in capital for τ going from 0·5 to 3·0

	$\sigma = 0{\cdot}01$	$\sigma = 0{\cdot}02$	$\sigma = 0{\cdot}03$
LQ	0·025%	0·049%	0·074%
log LQ	0·141%	0·510%	1·180%
PE	0·497%	1·854%	4·034%

4.3. *The accuracy test in the cash-in-advance model of Cooley and Hansen*

In this sub-section we use a monetary equilibrium model to study the properties of the accuracy test. In Cooley and Hansen (1989) a cash-in-advance model is solved by forming a LQ approximation. The standard LQ solution procedure, however, cannot be applied, since in a cash-in-advance economy the competitive equilibrium is not the solution to a planner's problem. Therefore, in addition to taking a LQ approximation of the utility function, the authors have to assume that the perceived law of motion for prices (scaled by the money supply) is linear in the state variables. This extension of the LQ solution technique is new to the literature and its accuracy has never been discussed. We tested the LQ solutions for accuracy and made comparisons with the solutions obtained with the method of parameterizing expectations (PE). More details are given in the Appendix, and in the working paper version of this article. Here we give a summary of the results.

The model has two stochastic Euler equations. One for capital and one for real money balances. We looked at two sets of parameters. The first one has a moderate money growth of 1·5% per quarter and the second has a high money growth of 15% per quarter.

With the PE numerical solution we can duplicate all the statistics that are reported in the tables in Cooley and Hansen (1989). Using the LQ solution we find that the results of the accuracy test for the money equation are dramatically bad for both parameter sets. The results for the residual of the Euler equation for capital are good for the low growth case but bad for the high growth case. The PE numerical solution, however, does pass the accuracy tests. The different results in the accuracy tests suggest that there are differences between the series generated by both methods, even though these do not affect the statistics reported by Cooley and Hansen and, therefore, the conclusions in that paper.

The main differences are as follows: (i) it is possible to show analytically that, in the true rational expectations equilibrium, the costs due to the cash-in-advance constraint (as a fraction of the marginal utility of consumption) are a fixed function of only the current money growth rate, so that the correlation between these two variables should be very close to one; with the LQ solution, however, we find a value of 0·939; (ii) the behaviour in the tails is different; (iii) the LQ value for the mean of capital stock is low: for the high money growth case it is 1·6% lower than the value obtained with PE, confirming our intuition in the previous sub-section of why the log LQ solution behaves differently.

This example illustrates that the accuracy test is very powerful, in the sense that it detects small inaccuracies. That is, the results of the accuracy test may reject the LQ solution, although only some characteristics of the LQ solution showed substantial differences with the (accurate) PE solution.

5. CONCLUDING REMARKS

Since the actual solution to rational expectations models is usually not known, it is useful to have criteria for judging the accuracy of the numerical solution. The test introduced in

this paper can be used to detect inaccurate solutions with low computing costs. The test can be used for guidance in any algorithm where one has to make choices about the functional form, the grid or the convergence criterion, and in sorting between alternative solutions. Furthermore, the test can be used to eliminate irrelevant terms from a high-order polynomial in the method of parameterized expectations, thereby reducing the computational costs of refining the solution.

Perhaps the most important message of this paper is that tests can be designed to check for accuracy in a way that is inexpensive, powerful and constructive.

APPENDIX: SOLUTIONS OF THE MODELS

In this Appendix we give more detailed information about the solutions to the used models. For details on the method of parameterized expectations we refer the reader to Marcet (1988) and den Haan and Marcet (1990). In Christiano (1990) a description of the LQ-approximation is presented.

TABLE A.1

β_f for the growth model of Section 3

	n=1	n=2	n=3
constant	2·0359	1·8106	1·8151
$\ln(k_{t-1})$	−0·4063	−0·3212	−0·3252
$\ln(\theta_t)$	−0·1157	−0·2243	−0·2747
$\ln(k_{t-1})^2$		−0·0152	−0·0130
$\ln(k_{t-1})$ in (θ_t)		0·0388	0·0725
$\ln(\theta_t)^2$		−0·0294	−0·0846
$\ln(k_{t-1})^3$			−0·0004
$\ln(k_{t-1})^2 \ln(\theta_t)$			0·0055
$\ln(k_{t-1}) \ln(\theta_t)^2$			0·0193
$\ln(\theta_t)^3$			−0·0117

The simple growth model

The first-order conditions for the simple growth model are (3.2) and

$$c_t^{-\tau} = \delta E_t c_{t+1}^{-\tau}(\alpha k_t^{\alpha-1} + \mu). \tag{A.1}$$

Let $P_n(k, \theta)$ stand for the nth-order linear polynomial. The conditional expectation in Equation (A.1) is parameterized with $\exp(P_n(\log(k), \log(\theta)))$, where $n = 1, 2, 3$. If $n = 1$, we get

$$c_t^{-\tau} = \beta_1 \exp(\beta_2 \log(k_{t-1}) + \beta_3 \log(\theta_t)). \tag{A.2}$$

The fixed point for the vector β is calculated using 29,000 observations. The solutions for the three polynomials are given in Table A.1. For the growth model of Section 4, the fixed-point parameters are given in Table A.2.

The LQ policy functions for $\tau = 3{\cdot}0$ and $0{\cdot}5$ are respectively given by

$$k_t = 0{\cdot}57631 + 0{\cdot}97967\, k_{t-1} + 2{\cdot}17301 \log(\theta_t) \tag{A.3}$$

$$k_t = 1{\cdot}55914 + 0{\cdot}94500\, k_{t-1} + 2{\cdot}54911 \log(\theta_t). \tag{A.3'}$$

For the log LQ approximation we get, respectively,

$$\log(k_t) = 0{\cdot}06799 + 0{\cdot}97967 \log(k_{t-1}) + 0{\cdot}07665 \log(\theta_t) \tag{A.4}$$

$$\log(k_t) = 0{\cdot}18395 + 0{\cdot}94500 \log(k_{t-1}) + 0{\cdot}08992 \log(\theta_t). \tag{A.4'}$$

TABLE A.2

β_f for the growth model of Section 4

	$\tau=0.5$		
	$\sigma=0.01$	$\sigma=0.02$	$\sigma=0.03$
constant	1.5953	1.8885	1.9982
$\ln(k_{t-1})$	−0.1229	−0.2238	−0.2575
$\ln(\theta_t)$	−0.2089	−0.2066	−0.2054
$\ln(k_{t-1})^2$	−0.0415	−0.0264	−0.0213
$\ln(k_{t-1})\ln(\theta_t)$	0.0323	0.0316	0.0312
$\ln(\theta_t)^2$	−0.0353	−0.0294	−0.0274
	$\tau=3.0$		
	$\sigma=0.01$	$\sigma=0.02$	$\sigma=0.03$
constant	0.6250	1.5387	2.1976
$\ln(k_{t-1})$	−0.0790	−0.6162	−0.8264
$\ln(\theta_t)$	−0.8616	−1.4209	−1.6521
$\ln(k_{t-1})^2$	−0.1573	−0.0767	−0.0450
$\ln(k_{t-1})\ln(\theta_t)$	−0.0692	0.0991	0.1694
$\ln(\theta_t)^2$	−0.2080	−0.1289	−0.1026

The Cooley–Hansen model

Let λ_t and η_t be the Lagrange multipliers of respectively the budget equation and the cash-in-advance constraint. The first-order conditions for the Cooley–Hansen model can be written as follows:

$$\lambda_t = E_t\delta\lambda_{t+1}(\theta_{t+1}\alpha k_t^{\alpha-1}h_{t+1}^{\alpha-1}+\mu), \tag{A.5}$$

$$\frac{\lambda_t}{p_t} = E_t\delta\frac{\lambda_{t+1}+\eta_{t+1}}{p_t}, \tag{A.6}$$

$$\frac{1}{c_t} = \lambda_t+\eta_t, \tag{A.7}$$

$$B = \lambda_t(1-\alpha)\theta_t k_{t-1}^{\alpha}h_t^{-\alpha}, \tag{A.8}$$

$$c_t+k_t+\frac{M_t-M_{t-1}}{p_t} = \theta_t k_{t-1}^{\alpha}h_t^{1-\alpha}+\mu k_{t-1}+(g-1)\frac{M_{t-1}}{p_t}, \tag{A.9}$$

$$p_t c_t = m_t. \tag{A.10}$$

If we multiply Equation (A.6) with M_t, impose the equilibrium condition and use Equation (A.10), then we get

$$\lambda_t c_t = \delta E_t\frac{M_t}{M_{t+1}} = \delta E_t\frac{1}{g_{t+1}}. \tag{A.11}$$

If we eliminate λ_t using Equation (A.7), then we get

$$1-\eta_t c_t = \delta E_t\left[\frac{1}{g_{t+1}}\right].$$

Since $E_t(1/g_{t+1})$ is a known function of g_t, we get the result mentioned in Section 4.2 that η_t as a fraction of marginal utility is a function of only g_t. This is a property that was strongly violated by the LQ solution. The conditional expectation in equation (A.11) can be calculated analytically, since g_t has a log-normal distribution. Thus we only have to parameterize the conditional expectation in equation (A.5). Let $h(k_{t-1}, \theta_t, g_t; \beta)$ be the parameterization for the conditional expectation, where β is the vector of parameters. Equation (A.1) is thus replaced by

$$\lambda_t = h(k_{t-1}, \theta_t, g_t; \beta) \tag{A.12}$$

Given initial values for the parameter of $h(\cdot)$, it is again easy to solve the model. Equation (A.12) directly solves for λ_t. Equation (A.11) can be used to solve for consumption and Equation (A.8) to solve for the labour supply. The budget equation solves for the capital stock. The cash-in-advance constraint together with the law

TABLE A.3

B_f for the Cooley–Hansen model

	$\bar{g}=1{\cdot}015$	$\bar{g}=1{\cdot}15$
constant	3·0275	2·9511
$\ln(k_{t-1})$	−0·2293	−0·2451
$\ln(\theta_t)$	−1·3177	−1·2734
$\ln(g_t)$	−0·0324	−0·0324
$\ln(k_{t-1})^2$	−0·0631	−0·0631
$\ln(k_{t-1})\ln(\theta_t)$	0·3553	0·3553
$\ln(\theta_t)^2$	−0·1833	−0·1833
$\ln(\theta_t)^3$	−1·3690	−1·3690

of motion for money gives the price level. Again we let $h(\cdot)$ be the exponential of a linear polynomial in the logs. The fixed point for the parameters of $h(\cdot)$ are calculated using 40,000 observations. The parameter values at the fixed point are reported in Table A.3.

List of Variables

c_t = consumption
k_t = end-of-period capital stock
θ_t = productivity
h_t = labour supply
m_t = end-of-period nominal money holdings
p_t = price level
M_t = nominal money supply
$g_t = M_t/M_{t-1}$
η_t = Lagrange multiplier of the cash-in-advance constraint
λ_t = Lagrange multiplier of the budget constraint

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REFERENCES

BROCK, W. A. and MIRMAN, L. J. (1972), "Optimal Economic Growth and Uncertainty: the Discounted Case", *Journal of Economic Theory*, **4**, 479–513.
CHRISTIANO, L. J. (1990), "Linear-Quadratic Approximation and Value-Function Iteration: A Comparison", *Journal of Business and Economic Statistics*, **8**, 99–113.
CHRISTIANO, L. J. and DEN HAAN, W. (1993), "Small Sample Properties of GMM for Business Cycle Analysis" (Working Paper, University of California at San Diego and Northwestern University).
COLEMAN, W. J. (1991), "Equilibrium in a Production Economy with an Income Tax", *Econometrica*, **51**, 1091–1104.
COOLEY, T.F. and HANSEN, G. D. (1989), "The Inflation Tax in a Real Business Cycle Model", *American Economic Review*, **79**, 733–748.
DEN HAAN, W. J. (1990a), "The Optimal Inflation Tax in a Sidrauski-type Model with Uncertainty", *Journal of Monetary Economics*, **25**, 389–409.
DEN HAAN, W. J. (1990b), "The Term Structure of Interest Rates in Real and Monetary Production Economies" (Unpublished Ph.D. Thesis, Carnegie Mellon University).
DEN HAAN, W. J. and MARCET, A. (1990), "Solving the Stochastic Growth Model by Parameterizing Expectations", *Journal of Business and Economic Statistics*, **8**, 31–34.
FAIR, R. C. and TAYLOR, J. B. (1983), "Solution and Maximum Likelihood Estimation of Dynamic Nonlinear Rational Expectations Models", *Econometrica*, **51**, 1169–1185.
HANSEN, L. P. (1982), "Large Sample Properties of Generalized Method of Moments Estimators", *Econometrica*, **50**, 1029–1054.
INGRAM, B. (1990), "Equilibrium Modelling of Asset Prices: Rationality versus Rules of Thumb", *Journal of Business and Economic Statistics*, **8**.
JUDD, K. L. (1989), "Minimum Weighted Residual Methods in Dynamic Economic Models" (manuscript, Hoover Institution).

KETTERER, J. and MARCET, A. (1989), "Introducing Derivative Securities in an Equilibrium Model" (manuscript, GSIA, Carnegie Mellon University).
KYDLAND, F. E. and PRESCOTT, E. C. (1982), "Time to Build and Aggregate Fluctuations", *Econometrica*, **50**, 1345–1370.
LEE, T., WHITE, H. and GRANGER, C. W. J. (1989), "Testing for Neglected Nonlinearities in Time Series Models" (manuscript, University of California).
MARCET, A. (1988), "Solution of Nonlinear Models by Parameterizing Expectations" (manuscript, GSIA, Carnegie Mellon University).
MARCET, A. and MARIMON, R. (1992), "Communication, Commitment and Growth", *Journal of Economic Theory*, **58**, 219–249.
MARCET, A. and MARSHALL, D. (1992), "Convergence of Approximate Solutions to Rational Expectations Equilibria using the Parameterized Expectations Solution Procedure" (Working paper, Universitat Pompeu Fabra and Northwestern University).
MARCET, A. and SINGLETON, K. J. (1989), "Equilibrium Asset Prices and Savings of Heterogeneous Agents in the Presence of Portfolio Constraints" (manuscript, G.S.I.A., Carnegie Mellon University).
MARSHALL, D. A. (1988), "Inflation and Asset Returns in a Monetary Economy" (Ph.D. thesis, GSIA, Carnegie Mellon University). (Forthcoming in *Journal of Finance*.)
MILLER, R. (1984), "Job Matching and Occupational Choice", *Journal of Political Economy*, **92**, 1086–1120.
NEWEY, W. K. and WEST, K. D. (1987), "A Simple Positive Semi-definite Heteroskedasticity and Autocorrelation Consistent Covariance Matrix", *Econometrica*, **55**, 703–708.
NOVALES, A. (1991), "Solving Non-linear Rational Expectations Models: A Stochastic Equilibrium Model of Interest Rates", *Econometrica*, **58**, 93–113.
RUST, J. (1987), "Optimal Replacement of GMC Bus Engines: An Empirical Model of Harold Zurcher", *Econometrica*, **55**, 999–1033.
SIMS, C. A. (1989), "Solving Nonlinear Stochastic Optimization and Equilibrium Problems Backwards" (Discussion Paper 15, Institute for Empirical Macroeconomics, Federal Reserve Bank of Minneapolis).
TAUCHEN, G. and HUSSEY, R. (1991), "Quadrature-Based Methods for Obtaining Approximate Solutions to Nonlinear Asset Pricing Models", *Econometrica*, **59**, 371–396.
TAYLOR, J. B. and UHLIG, H. (1990), "Solving Nonlinear Stochastic Growth Models: A Comparison of Alternative Solution Methods", *Journal of Business and Economic Statistics*, 8, 1–17.
WHITE, H. (1984) *Asymptotic Theory for Econometricians* (New York: Academic Press).
WOLPIN, K. I. (1984), "An Estimable Dynamic Stochastic Model of Fertility and Child Mortality", *Journal of Political Economy*, **92**, 852–873.